Lecture #10

Random Variables.

- Often when performing an experiment, we are interested in some function of the outcomes as opposed to the actual outcomes themselves.

- Ex: When rolling two dice, the sample space is $S = \{(a,b) : 1 \le a \le 6, 1 \le b \le 6\}$, and we may be interested in

: $S \to \mathbb{R} : (a,b) \mapsto a+b$

or $(a,b) \mapsto \max\{a,b\}$

or $(a,b) \mapsto \min\{a,b\}$

or $(a,b) \mapsto a \cdot b$ etc.

Ex: We flip a coin three times, $S = \{(i,j,k) : i,j,k \in \{H,T\}\}$

we may be interested in the # of heads in an outcome.

Such functions are called random variables.

- Since the value of a random variable is determined by the outcome of the experiment, we may assign probabilities to the possible values of a random var.

Ex: Rolling Two dice.

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

Sums to: 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

If X is the random variable $X(a,b) = a+b$, then

$P(X=2) = 1/36.$

$P(X=3) = 2/36.$

$P(X=4) = 3/36.$

$P(X=5) = 4/36$

$P(X=6) = 5/36.$

$P(X=7) = 6/36$

$P(X=8) = 5/36.$

$P(X=9) = 4/36.$

$P(X=10) = 3/36.$

$P(X=11) = 2/36.$

$P(X=12) = 1/36.$

Note that $\sum_{i=2}^{12} P(X=i) = 1.$

Ex: We have a well-shuffled deck of cards. Suppose that our experiment is drawing 4 cards with replacement. Let X = the number of aces in the draw.

- The possible values of X are $\{0, 1, 2, 3, 4\}$.

Lets find the probabilities for these values:

$P(X=0)$: → for each draw, there is a $\frac{48}{52}$ probability of not being an ace. If all draws are non-aces, then

$$P(X=0) = \left(\frac{48}{52}\right)\left(\frac{48}{52}\right)\left(\frac{48}{52}\right)\left(\frac{48}{52}\right) = \left(\frac{48}{52}\right)^4.$$

$P(X=i)$ → since there are 4 slots and i aces, there are $\binom{4}{i}$ many ways the cards can show up (i.e. AA--, A-A-, -AA-, ... for $i=2$).

$$\text{So } P(X=i) = \binom{4}{i}\underbrace{\left(\frac{4}{52}\right)^{i}}_{\text{number of aces}}\underbrace{\left(\frac{48}{52}\right)^{4-i}}_{\text{number of non aces}}$$

Ex: We roll a single die. Let X be the number of times we need to roll until we see a 1 (i.e $X=n$ if the n^{th} roll is a 1 and no previous roll was a 1).

What are the possible values of X?

$$X = 1, 2, 3, 4, 5, \dots$$

It is possible that you could roll a die 1000000000 times and never see a 1 (but unlikely).

$$P(X=i) = \underbrace{\left(\frac{5}{6}\right)^{i-1}}_{\text{the first } i-1 \text{ rolls are non-ones.}} \times \left(\frac{1}{6}\right)$$

the probability that the ith roll is a 1.

Question: What is the probability that $X \geq 2$?

Since each of the events are mutually exclusive,

$$P(X \geq 2) = \sum_{i=2}^{\infty} P(X=i)$$

$$= \sum_{i=2}^{\infty} \left(\frac{5}{6}\right)^{i-1}\left(\frac{1}{6}\right) = \left(\frac{1}{6}\right)\sum_{i=2}^{\infty}\left(\frac{5}{6}\right)^{i-1}$$

$$= \left(\frac{1}{6}\right)\sum_{j=1}^{\infty}\left(\frac{5}{6}\right)^{j}$$

$$= \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\sum_{j=1}^{\infty}\left(\frac{5}{6}\right)^{j-1}$$

$$= \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)\cdot\left(\frac{1}{1-\frac{5}{6}}\right) = \frac{5}{6}.$$

Equivalently, since we know that there is 100% chance that something happens (i.e. eventually, we will see a 1), and so $\sum_{i=1}^{\infty} P(X=i) = 1$.

So $P(X \geq 2) = 1 - \underbrace{P(X \leq 1)}_{=P(X=1).} = 1 - 1/6 = 5/6.$

Defn: For a random variable X, the function F defined by
$$F(x) = P(X \leq x) \quad , -\infty < x < \infty$$
is called a cumulative distribution function.

In the previous example,
$$F(x) = \sum_{i=1}^{x} \left(\frac{5}{6}\right)^{i-1}\left(\frac{1}{6}\right).$$

Discrete Random Variables.

A random variable that can only take countably many values is called a discrete random variable (DRV).

Defn: If X is a DRV, we define the <u>probability mass function</u> of X to be

$$p(x) := P(X = x).$$

The probability mass function is positive for at most countably many values of x. If X takes possible values $\{x_1, x_2, \ldots\}$, then

$$p(x_i) \geq 0 \quad \forall \{x_1, x_2, \ldots\}.$$

and $p(y) = 0$ for all other values.

Since every possible value of X is in $\{x_1, x_2, x_3 \ldots\}$ we have

$$\sum_{i=1}^{\infty} p(x_i) = 1.$$

<u>Expected Value of a DRV.</u>

Let X be a DRV with p.m.f. $f(x)$.

The expected value of X, written $E[X]$. or μ is defined to be the average of the possible values of X, weighted by their probabilities. That is

$$E[X] = \sum_{x : p(x) > 0} x\, p(x).$$